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On a Wave Equation with a Cubic Convolution

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We study the well-posedness of the Cauchy problem and the asymptotic behavior of solutions of the nonlinear wave equation $u_{tt} - \Delta u + m^2 + u(V * u^2) = 0$ in Euclidean space.

1. INTRODUCTION

Consider the equation

$$u_{tt} - \Delta u + m^2 u + u(V * u^2) = 0 \tag{1}$$

for $x \in \mathbb{R}^n$, $n \geqslant 3$, $m \geqslant 0$, where V is a real function and $*$ denotes spatial convolution. The stationary equation

$$-\Delta w + w(V * w^2) = \sigma w \tag{2}$$

is obtained by looking for separated solutions of (1), $u = \exp(i\lambda t)\, w(x)$, where $\sigma = \lambda^2 - m^2$. Equation (2) has been studied in [4–6], for instance. In case $V = -|x|^{-1}$, Eq. (2) was proposed by Hartree as a model for the helium atom. On the other hand, the time-dependent Schrödinger equation with interaction term $u(V * u^2)$ has recently been studied by Glassey [3], Ginibre and Velo [2] and Dias and Figueira [1]. This paper is devoted to the study of Eq. (1).

In Section 2 we study the Cauchy problem for (1) under the condition that $V \in L^{n/3} + L^\infty$. Even though the nonlinear term is not local, it does vanish

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where u vanishes and so system (1) is causal. We assume V is an even function so that energy is conserved.

In Section 3 we show that if $m > 0$ and V is $O(|x|^{-2})$, then *all* solutions of sufficiently small energy are scattering states. In order that all solutions, even large ones, exist for all time, we require V to be non-negative (Section 4). If $m > 0$, $V \geqslant 0$ and V is $O(|x|^{-2})$, we have a scattering operator defined on the space of *all* finite-energy solutions. On the other hand, if $m = 0$ and $|x|^3 V(x)$ is decreasing, then the energy of each solution propagates outward at speed one as $t \to \infty$.

2. Existence of Solutions

We use the standard notation $L^p = L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, for the Lebesgue spaces and $|\ |_p$ for the norm in L^p. Any integral sign to which no domain is attached will be understood to be taken over the whole Euclidean space $\mathbb{R}^n$. We denote by $H^k = H^k(\mathbb{R}^n)$ the usual Sobolev space, and by $\|\ \|_k$ the norm, of functions whose derivatives up to order k belong to L^2. We denote by $M^p = M^p(\mathbb{R}^n)$ the Marcinkiewicz space of measurable functions on $\mathbb{R}^n$ such that

$$\sup_{\lambda > 0} \{\lambda^p \operatorname{meas} [|u| > \lambda]\}^{1/p} < \infty,$$

$1 \leqslant p < \infty$. This space is sometimes denoted $L(p, \infty)$ and its elements are said to be of "weak type" L^p (see [9]). If I is an interval and X is a Banach space, we denote by $C(I; X)$ the space of strongly continuous functions from I to X.

Theorem 1. *Let $n \geqslant 3$ and let $V\colon \mathbb{R}^n \to \mathbb{R}$ belong to $L^{n/3} + L^\infty$. If $n \geqslant 4$, we allow $V \in M^{n/3} + L^\infty$. Let $u_0 \in H^1$ and $u_1 \in L^2$. Then there exists a maximal interval $I = (T_-, T_+)$ with $-\infty \leqslant T_- < 0 < T_+ \leqslant +\infty$ and a unique function $u \in C(I; H^1)$, $u^t \in C(I; L^2)$ satisfying Eq. (1) in I together with the initial conditions*

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x)$$

for $x \in \mathbb{R}^n$. If $T_+ < \infty$ $(-\infty < T_-)$, then $\|u(t)\|_1^2 + |u_t(t)|_2^2 \to \infty$ as $t \to T_+$ (as $t \to T_-$).

Proof. Write (1) as a system of two equations of first order in time. According to Segal [8] we need only prove that the operator $u \to^N u(V * u^2)$ is locally Lipschitz from H^1 into L^2, provided that $m > 0$. Let us write $V = V_1 + V_2$, where $V_1 \in L^{n/3}$ or $M^{n/3}$, $V_2 \in L^\infty$. By using Hölder's, Young's, and Sobolev's inequalities, we obtain

$$\begin{aligned}|u(V_1 * u^2)|_2 &\leqslant |u|_{2n/n-2} |V_1 * u^2|_n \\ &\leqslant |u|_{2n/n-2} |V_1|_{n/3} |u^2|_{n/n-2} \\ &= |V_1|_{n/3} |u|^3_{2n/n-2} \leqslant C |V_1|_{n/3} |\operatorname{grad} u|_2^3\end{aligned}$$

and

$$|u(V_2 * u^2)|_2 \leqslant |u|_2 |V_2|_\infty |u^2|_1 = |V_2|_\infty |u|_2^3.$$

In case $V \in M^{n/3}$, $|V_1|_{n/3}$ should be replaced by $|V_1|_{M^{n/3}}$. This should be understood throughout this paper.

It is clear that N is a cubic polynomial map from H^1 into L^2, that is, $N(u) = N(u, u, u)$, where $N(u_1, u_2, u_3) = u_1(V * (u_2 u_3))$ which is a bounded trilinear mapping from H^1 into L^2, and in particular is locally Lipschitz. In case $m = 0$ we need only write the equation as $u_{tt} - \Delta u + u = u - u(V * u^2)$ and consider the right side as the interaction. Since the mapping $u \to -u + u(V * u^2)$ also takes H^1 into L^2 and since it is locally Lipschitz, the proof is finished.

EXAMPLE. If $n \geqslant 4$ and

$$|V(x)| \leqslant c |x|^{-3}, \tag{3}$$

Theorem 1 is applicable, for such a function V belongs to $M^{n/3}$.

THEOREM 2. *Assuming furthermore that V is an even function, the unique solution satisfies the energy identity for $t \in I$:*

$$\frac{1}{2} \int (u_t^2 + |\operatorname{grad} u|^2 + m^2 u^2)\, dx$$

$$+ \frac{1}{4} \iint u^2(x, t)\, V(x - y)\, u^2(y, t)\, dx\, dy = \textit{constant}.$$

Proof. From Theorem 1 we know that

$$f = -u(V * u^2) \in C(I; L^2)$$

and u satisfies the equation $u_{tt} - \Delta u + m^2 u = f$. By linear theory it follows that

$$\frac{1}{2} \int (u_t^2 + |\operatorname{grad} u|^2 + m^2 u^2)\, dx \,\Big|_0^T = \int_0^T \int f u_t \, dx\, dt$$

for all $T \in I$. Now,

$$\int_0^T \int f u_t\, dx\, dt = -\frac{1}{2} \int_0^T (w_t,\ V * w)_{L^2}\, dt,$$

where $w = u^2 \in C(I; L^1 \cap L^{n/n-2})$ and $w_t = 2uu_t \in C(I; L^1 \cap L^{n/n-1})$. If $X = L^1 \cap L^{n/n-2}$ and $Y = L^1 \cap L^{n/n-1}$, then $X \subseteq Y$, $w \in C(I; X)$ and $w_t \in C^1(I; Y)$. Let B be the operator $Bf = V * f$. Since $V \in L^{n/3} + L^\infty$ or $M^{n/3} + L^\infty$, it follows that B is a continuous linear operator from X into Y^* (the dual of Y). Since V is even, $B^*f = Bf$ $\forall f \in X$. These properties imply that (w, Bw) is of class C^1 and

$$\frac{d}{dt}(w, Bw) = (w_t, Bw) + (w, Bw_t) = 2(w_t, Bw).$$

Therefore we obtain

$$\int_0^T \int fu_t \, dx \, dt = -\frac{1}{4} \int_0^T \frac{d}{dt}(w, V * w) \, dt$$

$$= -\frac{1}{4}(u^2, V * u^2) \Big|_0^T.$$

This completes the proof of Theorem 2.

COROLLARY. *If V is even and $u(x,t)$ and $v(x,t)$ are two solutions as in Theorem* 1, *then there exists a constant C, depending only on the energies of u and v such that*

$$\|u(t) - v(t)\|_e \leqslant \|u(0) - v(0)\|_e e^{Ct}, \text{ where}$$

$$\|y(t)\|_e = \left\{ \int (y_t^2 + |\text{grad } y|^2 + m^2 y^2) \, dx \right\}^{1/2}. \tag{4}$$

Proof. $u - v$ satisfies the differential equation with the nonlinear term $N(u) - N(v) = u(V * u^2) - v(V * v^2)$. Therefore

$$\|u(t) - v(t)\|_e - \|u(0) - v(0)\|_e$$

$$\leqslant \int_0^t |N(u(\tau) - N(v(\tau))|_2 \, d\tau$$

$$\leqslant \text{Const} \int_0^t (\|u(\tau)\|_e + \|v(\tau)\|_e)^2 \, \|u(\tau) - v(\tau)\|_e \, dt$$

$$\leqslant \text{Const} \int_0^t \|u(\tau) - v(\tau)\|_e \, d\tau$$

by Theorem 2. Then we use Gronwall's inequality.

THEOREM 3 (Causality). *Under the conditions of Theorem* 1, *the values of the solution in* $\{t=t_0, |x-x_0|<\delta\}$ *depend only on the values of the initial data in* $\{t=0, |x-x_0|<\delta+|t_0|\}$, *provided* t_0 *belongs to the interval of existence and* δ *is a positive number.*

Proof. We will prove the equivalent statement: if the initial data vanish at $\{t=0, |x-x_0|<R\}$, then the solution vanishes in the double cone $\{|x-x_0|<R-|t|, t\in I\}$. Using Theorem 4 we may assume the solution is smooth. For convenience we consider only positive times. The solution has bounded energy norm in $0\leqslant t\leqslant T_1$, if $[0,T_1]\subseteq I$. Multiplying the equation by u_t, we obtain the identity

$$\frac{1}{2}\frac{\partial}{\partial t}[u_t^2+|\operatorname{grad} u|^2+m^2u^2]-\operatorname{div}[u_t\operatorname{grad} u]=-u_tu(V*u^2).$$

Let C_T be the piece of solid cone

$$\{(x,t), |x-x_0|<R-t, 0<t<T\},$$

where $T<T_1$. Let $\Omega_t=\{x, |x-x_0|<R-t\}$ be its cross section at time t. Integrating the identity over C_T and using the divergence theorem, we have

$$e(T)-e(0)+\frac{1}{\sqrt{2}}\int_{K_T}\left[\frac{1}{2}(u_t^2+|\operatorname{grad} u|^2+m^2u^2)-u_t(\nu\cdot\operatorname{grad} u)\right]dS$$
$$=-\int_0^T\int_{\Omega_t}u_tu(V*u^2)\,dx\,dt,$$

where

$$e(t)=\frac{1}{2}\int_{\Omega_t}(u_t^2+|\operatorname{grad} u|^2+m^2u^2)\,dx,$$

K_T is the lateral boundary of C_T and ν is the unit normal to Ω_t. In this identity we shall drop the integral over K_T since its integrand is non-negative. We bound the right side as in Theorem 1 to obtain

$$\left|\int_{\Omega_t}u_tu(V*u^2)\,dx\right|$$
$$\leqslant\left(\int_{\Omega_t}u_t^2\,dx\right)^{1/2}\left(\int_{\Omega_t}u^2(V*u^2)^2\,dx\right)^{1/2}$$

$$\leqslant \left(\int_{\Omega_t} u_t^2\,dx\right)^{1/2} \left[\left(\int_{\Omega_t} |u|^{2n/(n-2)}\,dx\right)^{(n-2)/2n} |V_1|_{n/3}\,|u|_{2n/n-2}^2 + \left(\int_{\Omega_t} u^2\,dx\right)^{1/2} |V_2|_\infty\,|u|_2^2\right]$$

$$\leqslant ce(t).$$

Therefore $e(T)-e(0)\leqslant C\int_0^T e(t)\,dt$, where C is independent of T, provided $0\leqslant T\leqslant T_1$. Hence

$$e(t)\leqslant e(0)\,e^{Ct} \qquad \text{for} \quad 0\leqslant t\leqslant T_1.$$

So, if the initial data vanishes in $\{|x-x_0|<R\}$, then the solution vanishes in $\{(x,t),\ |x-x_0|<R-t,\ 0\leqslant t\leqslant T_1\}$. This completes the proof if $m>0$. In case $m=0$ we use the same device as in Theorem 1, regarding $u(V*u^2)-u$ as the nonlinear term and noting that

$$\left|\int_{\Omega_t} u_t u\,dx\right| \leqslant ce(t)\equiv c\int_{\Omega_t} (u_t^2+|\operatorname{grad} u|^2+u^2)\,dx.$$

THEOREM 4 (Regularity). *Let V satisfy the conditions of Theorem 1. Let k be a non-negative integer. If $u_0\in H^{k+1}$ and $u_1\in H^k$, then $u\in C(I;H^{k+1})$ and $u_t\in C(I;H^k)$.*

Proof. As in the proof of Theorem 1, we write (1) as a first-order system on the Hilbert space $H^1\oplus L^2$:

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t\end{bmatrix} = \begin{bmatrix} 0 & I \\ m^2-\Delta & 0\end{bmatrix}\begin{bmatrix} u \\ u_t\end{bmatrix} + \begin{bmatrix} 0 \\ N(u)\end{bmatrix} \equiv Au+\mathcal{N}(u).$$

Following Segal [8] or Reed [7], we only need to verify that the pair $[u_0,u_1]$ belongs to $D(A^k)$ and that the nonlinear term $\mathcal{N}$ carries $D(A^k)$ into $D(A^k)$ with appropriate bounds. Now $D(A^k)=H^{k+1}\oplus H^k$, assuming for the time being that $m>0$. So it remains to show that $N\colon H^{k+1}\to H^k$ with appropriate Lipschitz bounds. We now sketch the proof of this fact for $k=1$. We must estimate the L^2 norm of $\partial_j[(V*u^2)u]$, where $\partial_j=\partial/\partial x_j$. It is the sum of two terms. The first one is

$$\begin{aligned}|(V*u^2)\,\partial_j u|_2 &\leqslant |V_1*u^2|_n\,|\partial_j u|_{2n/(n-2)} + |V_2*u^2|_\infty\,|\partial_j u|_2\\ &\leqslant c\,|V_1|_{n/3}\,|\operatorname{grad} u|_2^2\,|\Delta u|_2 + |V_2|_\infty\,|u|_2^2\,|\partial_j u|_2\\ &\leqslant c(V)\,\|u\|_2^3.\end{aligned}$$

The other term is

$$\begin{aligned}
&|(V * u\,\partial_j u)u|_2 \\
&\quad \leqslant |V_1 * u\,\partial_j u|_n\,|u|_{2n/(n-2)} + |V_2 * u\,\partial_j u|_\infty\,|u|_2 \\
&\quad \leqslant c\,|V_1|_{n/3}\,|\partial_j u|_{2n/(n-2)}\,|u|^2_{2n/(n-2)} + |V_2|_\infty\,|\partial_j u|_2\,|u|_2^2 \\
&\quad \leqslant c(V)\,\|u\|_2^3.
\end{aligned}$$

It is clear from these estimates that $N(u) - N(v)$ can be similarly treated in terms of u, v and $(u - v)$, and that derivatives of arbitrary order can be treated. We omit the details.

3. Solutions of Low Energy

THEOREM 5. *Let $n \geqslant 3$ and $m > 0$. Let V be an even, measurable function such that*

$$|V(x)| \leqslant c/|x|^2$$

for all $x \neq 0$. Then there exists $\delta > 0$ with the following property. (a) *For arbitrary initial data $[u_0, u_1]$ such that*

$$\frac{1}{2}\int (u_1^2 + |\operatorname{grad} u_0|^2 + m^2 u_0^2)\,dx < \delta,$$

the unique solution u exists for all time (that is, $I = (-\infty, +\infty)$ in Theorem 1). (b) *Furthermore, there exists a unique pair of solutions u_+ and u_- of the "free equation" (that is,* (1) *with $V = 0$) such that*

$$\|u(t) - u_\pm(t)\|_e \to 0 \qquad as \quad t \to \pm\infty. \tag{5$_\pm$}$$

(c) *The local energy of u tends to zero,*

$$\int_{|x| < (1-\epsilon)|t|} (u_t^2 + |\operatorname{grad} u|^2 + m^2 u^2)\,dx \to 0 \tag{6}$$

as $t \to \pm\infty$ for $\varepsilon > 0$.

THEOREM 6. *Under the same conditions as in Theorem* 5, *let u_- be a free solution of energy less than δ.* (a) *There exists a unique solution u of Eq.* (1) *for all time such that* $(5)_-$ *holds.* (b) *Furthermore, there exists a unique free solution u_+ such that* $(5)_+$ *holds.* (c) *The solution u satisfies* (6). (d) *The energy norm of u_+ equals the energy norm of u_-. The scattering operator S, which carries u_- to u_+ is a homeomorphism in the energy norm.*

Proofs. We apply the method of [11], choosing $X = H^1 \oplus L^2$ provided with the energy norm, $X_1 = \{0\} + L^\mu$ and $X_3 = L^q \oplus W^{-1,q}$, where $\mu = 2n/(n+1)$ and $q = 2n/(n-1)$. Here $W^{k,q}$ denotes the usual Sobolev space with k derivatives in L^q. The method we use requires various properties of the "free equation" with respect to the spaces X_1 and X_3. The hypotheses which relate to the nonlinear equation (1) are as follows. Hypothesis II requires $\mathscr{N}: X_3 \to X_1$, or

$$|N(f) - N(g)|_\mu \leqslant c(|f|_q + |g|_q)^2 \, |f - g|_q^2.$$

Indeed, since $V \in M^{n/2}$,

$$\begin{aligned} |N(f) - N(g)|_\mu &\leqslant |(f-g)(V * f^2)|_\mu + |g(V * (f^2 - g^2))|_\mu \\ &\leqslant |f-g|_q \, |V * f^2|_n + |g|_q \, |V * (f^2 - g^2)|_n \\ &\leqslant c \, |f-g|_q \, |f^2|_{n/(n-1)} + c \, |g|_q \, |f^2 - g^2|_{n/(n-1)} \end{aligned}$$

which implies II. Hypothesis VI is satisfied if the functional

$$G(f) = \tfrac{1}{4}(f^2, V * f^2)$$

is continuous on L^q. In fact,

$$|G(f)| \leqslant |f^2|_{n/(n-1)} \, |V * f^2|_n \leqslant c \, |f|_q^4$$

and similarly G is locally Lipschitz. Hypothesis VII is an immediate consequence of Theorems 1 and 2. Finally we apply Theorem 16 of [11] with $p = 3$, $r = 4$ and $d = 1/2$. The local energy decay of "free solutions" together with $(5)_\pm$ immediately implies (6).

Remark 1. The same results are valid more generally if $n/2 \geqslant z \geqslant (n+1)/4$ and either $V \in M^z$, $z > 1$ or else $V \in L^z$, $z \geqslant 1$.

Remark 2. If we weaken the assumption on the initial data $[u_0, u_1]$ in Theorem 5 and on the incident free solution, u in Theorem 6 to say that a certain Sobolev norm is less than δ, then we can allow arbitrary $V \in L^z + L^1$, where $3n/4 > z > 1$. S is no longer a homeomorphism in the energy norm. Further details are given in [11].

4. Solutions of High Energy

THEOREM 7. *Let V be an even function satisfying the conditions of Theorem 1. If $V(x) < 0$ near the origin, then there exist initial data for which the interval I of existence (in Theorem 1) is finite.*

Proof. If $V(x) < 0$ for $|x| < a$, let $\psi(x)$ have support in $|x| < a/2$, so that $(\psi^2, V * \psi^2) < 0$. Let $\lambda > 0$ and $u_0(x) = \lambda\psi(x)$. Let $u_1(x)$ be any function such that $\int u_1(x)\,\psi(x)\,dx > 0$. Then

$$E = \frac{1}{2}\int u_1^2\,dx + \frac{\lambda^2}{2}\int (|\mathrm{grad}\,\psi|^2 + m^2\psi^2)\,dx$$
$$+ \frac{\lambda^4}{4}(\psi^2, V * \psi^2) < 0$$

for λ sufficiently large. Let $u(x, t)$ be the solution of (1) with initial data $\{u_0, u_1\}$. Now we multiply Eq. (1) by u and integrate to obtain

$$g''(t) + \int [-u_t^2 + |\nabla u|^2 + m^2u^2 + u^2(V * u^2)]\,dx = 0,$$

where $g(t) = \frac{1}{2}\int u^2(x, t)\,dx$. To this identity we subtract $4E$, using Theorem 2:

$$g''(t) - \int [3u_t^2 + |\nabla u|^2 + m^2u^2]\,dx = -4E.$$

Since $E \leqslant 0$,

$$g''(t)\,g(t) \geqslant \frac{3}{2}\int u_t^2\,dx \int u^2\,dx \geqslant \frac{3}{2}[g'(t)]^2.$$

Therefore $h(t) \equiv [g(t)]^{-1/2}$ is concave and $h(0) > 0$, $h'(0) < 0$. Hence $h(T) = 0$ for some $T > 0$ and so $g(t) \to \infty$ as $t \nearrow T$, unless the solution does not exist up to time T.

Because of Theorem 7 we assume from now on that $V \geqslant 0$. In this case all solutions are global.

THEOREM 8. *If V is a non-negative, even function and the conditions of Theorem* 1 *hold, then $I = (-\infty, \infty)$, that is, the solution exists for all time.*

Proof. Let $m > 0$. By Theorem 2 and the positivity of V, the energy norm of u is bounded a priori. According to the last statement of Theorem 1, T_0 could not be finite. Now let $m = 0$. As above, $|\mathrm{grad}\,u|_2^2 + |u_t|_2^2$ is bounded. But $|u(t)|_2 \leqslant |u(0)|_2 + ct$. Hence, T_0 could not be finite in this case either.

THEOREM 9 (Weak scattering). *Let m, n and V satisfy the conditions of Theorem* 5. *In addition, let $V \geqslant 0$. Let u_- be an arbitrary free solution of finite energy (not necessarily small). Then Part* (a) *of Theorem* 6 *is valid.*

Furthermore, there exists a unique free solution u_+ *which is weakly asymptotic to* u *as* $t \to +\infty$. [*This means that*

$$((u(t), \varphi(t))) \to ((u_+(0), \varphi(0))) \qquad \text{as} \quad t \to +\infty \tag{7}$$

for all free solutions φ *of finite energy, where* $((\ ,\ ))$ *denotes the energy inner product.*] *The energy of* u_+ *is no greater than that of* u_-.

Proof. The first statement follows from Theorem 3 of [11]. The second follows from the fact that $V \geqslant 0$, so that the energy norm of u is bounded (see Theorem 4.2 of [10]). The remarks following Theorems 5 and 6 are applicable here as well.

Now we use a completely different argument if $m = 0$ to prove that the local energy decays.

THEOREM 10. *Let* $m = 0$ *and* $n \geqslant 3$. *Let* V *be a non-negative, even,* C^1 *function such that* $|x|^3 V(x)$ *is non-increasing in* $|x|$. *Let* $u(x, t)$ *be a solution* (*as in Theorem* 8) *with initial data of compact support. Then*

$$\int_{|x| < (1-\epsilon)|t|} (u_t^2 + |\text{grad}\, u|^2)\, dx \to 0$$

as $t \to \pm\infty$ *for any* $\varepsilon > 0$.

Proof. Let $r = |x|$. The assumption on V means that $(\partial/\partial r)(r^3 V) \leqslant 0$ or

$$rV_r + 3V \leqslant 0, \tag{8}$$

where $\partial/\partial r = (x/r) \cdot \text{grad}$. It obviously implies (3). Theorems 1, 2, 3 and 8 are applicable. Let the supports of u_0 and u_1 be contained in $\{x, |x| \leqslant k\}$. By Theorem 3, the support of u is contained in $\{(x, t), |x| \leqslant |t| + k\}$. Suppose first that u_0 and u_1 are C^∞ functions. By Theorem 4, u is C^∞. The well-known dilation identity [10] is

$$\begin{aligned}
&[u_{tt} - \Delta u] \left[tu_t + ru_r + \frac{(n-1)}{2} u \right] \\
&\quad = \left[t \left(\frac{1}{2} u_t^2 + \frac{1}{2} |\text{grad}\, u|^2 \right) + \left(ru_r + \frac{(n-1)}{2} u \right) u_t \right]_t \\
&\qquad + \nabla \cdot \left[-tu_t\, \text{grad}\, u - \frac{1}{2} xu_t^2 \right. \\
&\qquad \left. + \frac{1}{2} x\, |\text{grad}\, u|^2 - ru_r\, \text{grad}\, u - \frac{(n-1)}{2} u\, \text{grad}\, u \right].
\end{aligned}$$

Furthermore

$$[(V*u^2)u]\left[tu_t+ru_r+\frac{(n-1)}{2}u\right]$$
$$=\left[\frac{1}{4}t(V*u^2)u^2\right]_t+\nabla\cdot\left[\frac{1}{2}x(V*u^2)u^2\right]$$
$$-\frac{1}{2}\sum_{j=1}^{n}x_j\left(\frac{\partial V}{\partial x_j}*u^2\right)u^2-\frac{3}{4}(V*u^2)u^2.$$

Because $\partial V/\partial x_j$ is an odd function,

$$\int x_j\left(\frac{\partial V}{\partial x_j}*u^2\right)u^2\,dx=\frac{1}{2}\int\left[\left(x_j\frac{\partial V}{\partial x_j}\right)*u^2\right]u^2\,dx.$$

Adding and integrating, we have

$$0=\frac{d}{dt}\int\left\{t\left[\frac{1}{2}u_t^2+\frac{1}{2}|\operatorname{grad}u|^2+\frac{1}{4}(V*u^2)u^2\right]\right.$$
$$\left.+\left(ru_r+\frac{(n-1)}{2}u\right)u_t\right\}dx-\frac{1}{4}\int([rV_r+3V]*u^2)u^2\,dx.$$

Integrating over time and using (8), we have

$$\frac{1}{2}T\int\left(u_t^2+|\operatorname{grad}u|^2+\frac{1}{2}(V*u^2)u^2\right)\Big|_{t=T}dx$$
$$+\int\left[\left(ru_r+\frac{(n-1)}{2}u\right)u_t\right]_{t=T}dx$$
$$\leqslant\int\left(r\frac{\partial u_0}{\partial r}+\frac{(n-1)}{2}u_0\right)u_1\,dx \tag{9}$$

for any $T>0$. It is convenient to denote by $\mathbf{w}$ the vector field

$$\mathbf{w}=\operatorname{grad}u+\frac{(n-1)}{2}\frac{\mathbf{x}}{r^2}u.$$

Then

$$|\mathbf{w}|^2=|\operatorname{grad}u|^2-\frac{(n-1)(n-3)}{4}\frac{u^2}{r}+\nabla\cdot\left[\frac{(n-1)}{2}\frac{\mathbf{x}}{r^2}u^2\right]. \tag{10}$$

Denote by I the first two integrals in (9). By (10) we can write $I=J+K$, where

$$J=\int\left[(T+k)\left(\tfrac{1}{2}u_t^2+\tfrac{1}{2}|\mathbf{w}|^2\right)+\mathbf{x}\cdot\mathbf{w}u_t\right]dx$$

and

$$K=-\frac{k}{2}\int(u_t^2+|\mathbf{w}|^2)\,dx+\frac{T(n-1)(n-3)}{8}\int\frac{u^2}{r^2}\,dx$$
$$+\frac{T}{4}\int(V*u^2)u^2\,dx.$$

Because $u(x,T)$ has support in $\{|x|<T+k\}$, the integrand in J is non-negative. So we get a lower bound for J by throwing away the part of it over $\{|x|>(1-\varepsilon)T\}$:

$$J\geqslant\frac{1}{2}\varepsilon T\int_{|x|<(1-\epsilon)T}(u_t^2+|\mathbf{w}|^2)\,dx.$$

In this integral and in K we again use (10). From (9) we obtain

$$\frac{1}{2}\varepsilon T\int_{|x|<(1-\epsilon)T}(u_t^2+|\operatorname{grad}u|^2)\,dx-\frac{k}{2}\int(u_t^2+|\operatorname{grad}u|^2)\,dx$$
$$+[(1-\varepsilon)T-k]\frac{(n-1(n-3)}{8}\int\frac{u^2}{r^2}\,dx+\varepsilon T\frac{(n-1)}{4}$$
$$\times\int_{|x|=(1-\epsilon)T}\frac{u^2}{r}\,dS+\frac{T}{4}\int(V*u^2)u^2\,dx\leqslant\int\left(r\frac{\partial u_0}{\partial r}+\frac{n-1}{2}u_0\right)dx. \tag{11}$$

Each term on the left side of (11) is non-negative, except for the second integral which is bounded (by Theorem 2). Therefore each term in (11) is bounded for all time. From the first term we get the desired result.

In case u_0 and u_1 are not C^∞, we approximate them by C^∞ functions with bounded support $u_{0\epsilon}$ and $u_{1\epsilon}$ (in H^1 and L^2, respectively). Let u_ϵ be the solution for the initial data $u_{0\epsilon}$ and $u_{1\epsilon}$. By the corollary to Theorem 2, each term in (11) converges as $\varepsilon\to 0$. Hence (11) is valid for all solutions with $u_0\in H^1$, $u_1\in L^2$ with compact support in $|x|\leqslant k$. This completes the proof of the theorem.

COROLLARY.

$$\int(V*u^2)u^2\,dx=O(t^{-1})\qquad as\quad t\to\infty.$$

Proof. Immediate from inequality (11).

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